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AN ENERGY IDENTITY IN PHYSICALLY NONLINEAR ELASTICITY AND ERROR ESTIMATES OF THE PLATE EQUATIONS

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An identity generalizing the Prager-Synge relationship [1, 2] in linear elasticity is deduced for a certain class of nonlinear elasticity laws. It permits estimation of the energy norm of the difference between some statically admissible stress field σ and the true field σ°, as well as between some kinematically admissible displacement field $\mathbf{u}$ and the true field $\mathbf{u}^\circ$, in terms of the energy norm for the difference between the fields σ and $\sigma(\mathbf{u})$ ($\sigma(\mathbf{u})$ is the stress field generated by the field $\mathbf{u}$). By using this identity, under definite constraints, it is proved that the root-mean-square value (over the volume of a plate) of the error in the solution of the plate equations derived from the volume problem by means of the Kirchhoff hypothesis, does not exceed $ch^{1/2}$, where c is a constant and h is the relative thickness. The Prager-Synge relationship [1, 2] was used in [3, 4] to estimate the error in linear shell theory. The results are related to [1 — 7].

1. Let the specific strain energy be [8]

$$A(\boldsymbol{\varepsilon}) = {}^9/_2\, K (\varepsilon_0)^2 + A_2(\psi_\varepsilon) \tag{1.1}$$
$$\boldsymbol{\varepsilon} = \{\varepsilon_{ij}\}, \quad \varepsilon_0 \equiv {}^1/_3\, (\varepsilon_{ii}), \quad \psi_\varepsilon \equiv (2^{-1}\varepsilon_{ij}'\varepsilon_{ij}')^{1/2}, \quad \varepsilon_{ij}' \equiv \varepsilon_{ij} - \delta_i^j \varepsilon_0$$

Here $\boldsymbol{\varepsilon}$ is the strain tensor, ε_0 is the mean elongation, ψ_ε is the shear strain intensity, δ_i^j is the Kronecker delta, K is the modulus of volume expansion, $A_2(\tau)$ is a function given for $\tau \in [0, \infty)$ which is twice differentiable and satisfies the conditions

$$A_2 = dA_2 / d\tau = 0, \quad \tau = 0, \quad 4G_1 \leqslant d^2A_2 / d\tau^2 \leqslant 4G_2 \tag{1.2}$$
$$\forall \tau \in [0, \infty)$$

$G_1, G_2 > 0$ are constants constraining the limits of variation of the shear modulus.

Let us introduce the notation

$$(\mathbf{a}, \mathbf{b}) \equiv a_{ij}b_{ij}, \quad |\mathbf{a}| \equiv (\mathbf{a}, \mathbf{a})^{1/2} \quad (\mathbf{a} = \{a_{ij}\}, \ \mathbf{b} = \{b_{ij}\})$$

Lemma 1. Under the conditions (1.2) the function $A(\boldsymbol{\varepsilon})$ is strictly convex, and moreover, for any $\boldsymbol{\varepsilon} = \{\varepsilon_{ij}\}$, $\boldsymbol{\varepsilon}^1 = \{\varepsilon_{ij}^1\}$ the estimates

$$\gamma |\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^1|^2 \leqslant (\nabla A(\boldsymbol{\varepsilon}) - \nabla A(\boldsymbol{\varepsilon}^1), \ \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^1) \leqslant \varkappa |\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^1|^2 \tag{1.3}$$
$$\nabla A(\boldsymbol{\varepsilon}) \equiv \{\partial A(\boldsymbol{\varepsilon}) / \partial \varepsilon_{ij}\}, \quad \gamma = \min\{3K, \ 2G_1\}$$
$$\varkappa = \max\{3K, 2G_2\}$$
$$\gamma |\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^1| \leqslant |\nabla A(\boldsymbol{\varepsilon}) - \nabla A(\boldsymbol{\varepsilon}^1)| \leqslant \varkappa |\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^1| \tag{1.4}$$

are valid.

Let an elastic body occupy a domain V of the variables (x_1, x_2, x_3) and let S be the piecewise-smooth boundary of the domain V. We consider the following elasticity problem:

$$\mathbf{u} = (u_1, u_2, u_3), \quad \varepsilon_{ij}(\mathbf{u}) = 2^{-1}(u_{i,j} + u_{j,i}), \quad \boldsymbol{\varepsilon}(\mathbf{u}) = \{\varepsilon_{ij}(\mathbf{u})\} \tag{1.5}$$
$$\boldsymbol{\sigma} = \{\sigma_{ij}\} = \nabla A(\boldsymbol{\varepsilon}) \tag{1.6}$$
$$\sigma_{ij,j} + f_i = 0, \quad i = 1, 2, 3 \tag{1.7}$$
$$\sigma_{ij}n_j = F_i, \quad i = 1, 2, 3 \quad \text{on } S_F \tag{1.8}$$
$$u_i = 0, \quad i = 1, 2, 3 \quad \text{on } S_u, \qquad S = S_F \cup S_u \tag{1.9}$$

From (1.3) the elasticity law (1.6) is uniquely reversible; it is known from [9] that an inverse mapping is given by using the dual function $A^*(\boldsymbol{\sigma})$

$$\boldsymbol{\varepsilon} = \nabla A^*(\boldsymbol{\sigma}) \equiv \{\partial A^*(\boldsymbol{\sigma}) / \partial \sigma_{ij}\} \tag{1.10}$$

where if $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are connected by the relationship (1.6) or (1.10), then

$$A^*(\boldsymbol{\sigma}) + A(\boldsymbol{\varepsilon}) = (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \tag{1.11}$$

It can be verified that

$$A^*(\boldsymbol{\sigma}) = (2K)^{-1}(\sigma_0)^2 + A_2^*(2\psi_\sigma)$$
$$\sigma_0 \equiv {}^1/_3\, (\sigma_{ii}), \quad \psi_\sigma \equiv (2^{-1}\sigma_{ij}'\sigma_{ij}')^{1/2}, \quad \sigma_{ij}' \equiv \sigma_{ij} - \delta_i^j \sigma_0$$

(here $A_2^*(t)$ is the dual function of $A_2(\tau)$) and the following inequalities are valid for any σ, σ^1

$$\varkappa^{-1} |\boldsymbol{\sigma} - \boldsymbol{\sigma}^1|^2 \leqslant (\nabla A^*(\boldsymbol{\sigma}) - \nabla A^*(\boldsymbol{\sigma}^1), \ \boldsymbol{\sigma} - \boldsymbol{\sigma}^1) \leqslant \gamma^{-1} |\boldsymbol{\sigma} - \boldsymbol{\sigma}^1|^2 \tag{1.12}$$
$$\varkappa^{-1} |\boldsymbol{\sigma} - \boldsymbol{\sigma}^1| \leqslant |\nabla A^*(\boldsymbol{\sigma}) - \nabla A^*(\boldsymbol{\sigma}^1)| \leqslant \gamma^{-1} |\boldsymbol{\sigma} - \boldsymbol{\sigma}^1| \tag{1.13}$$

Let $\varepsilon \in L_2(V)$ be the tensor field, then from (1.4),(1.13) the elasticity law (1.6) is a one-to-one conformal continuous mapping comparing the field $\sigma \in L_2(V)$ to every $\varepsilon \in L_2(V)$; the inverse mapping (1.10) is also continuous.

Theorem 1 [10—12] (minimum total energy principle). Let

$$f_i \in L_2(V), \quad F_i \in L_2(S_F), \quad i=1,2,3 \tag{1.14}$$

We introduce the space H of admissible displacement fields and the total energy functional

$$H = \{\mathbf{u} \mid \mathbf{u} = (u_1, u_2, u_3),\ u_i \in W_2^1(V),\ u_i = 0 \text{ on } S_u,\ i = 1, 2, 3\} \tag{1.15}$$

$$\Phi(\mathbf{u}) = \int_V A(\varepsilon(\mathbf{u}))\,dV - \int_V f_i u_i\,dV - \int_{S_F} F_i u_i\,dS \tag{1.16}$$

Then there exists a single field $\mathbf{u}^0$ providing the minimum of the functional $\Phi(\mathbf{u})$ in the space H.

We assume

$$\varepsilon^\circ \equiv \varepsilon(\mathbf{u}^\circ), \quad \sigma^\circ \equiv \nabla A(\varepsilon^\circ) \tag{1.17}$$

then for any field $\mathbf{u} \in H$ the identity

$$\int_V (\sigma^\circ, \varepsilon(\mathbf{u}))\,dV = \int_V f_i u_i\,dV + \int_{S_F} F_i u_i\,dS \tag{1.18}$$

is valid.

Definition 1. We call the field $\sigma = \{\sigma_{ij}\}$ statically admissible if $\sigma \in L_2(V)$, $\sigma_{ij} = \sigma_{ji}$ and the identity

$$\int_V (\sigma, \varepsilon(\mathbf{u}))\,dV = \int_V f_i u_i\,dV + \int_{S_F} F_i u_i\,dS \tag{1.19}$$

is valid for any $\mathbf{u} \in H$.

The set of statically admissible fields will be denoted by P and $\sigma^\circ \in P$ from (1.18).

Note 1. If the field σ is statically admissible in the ordinary sense, i.e. σ_{ij} are differentiable and satisfy the equilibrium equations (1.7) and the boundary conditions (1.8), then σ is statically admissible in the sense of the Definition 1.

Lemma 2. The set P is convex and closed in $L_2(V)$.

Proof. The convexity of P is evident, let us prove it closed, i.e. we prove that if the sequence $\sigma^n = \{\sigma_{ij}^n\}$ belongs to P and converges to some field $\sigma^* = \{\sigma_{ij}\}$ in $L_2(V)$ as $n \to \infty$, then $\sigma^* \in P$. Since $\sigma^n \in P$, then $\sigma_{ij}^n = \sigma_{ji}^n$ and from the condition $\| \sigma^n - \sigma^* \|_{L_2(V)} \to 0$ there follows that $\sigma_{ij}^* = \sigma_{ji}^*$.

We substitute σ^n into (1.19) and pass to the limit as $n \to \infty$ for fixed $\mathbf{u}$, we obtain that σ^* satisfies the identity (1.19), i.e. $\sigma^* \in P$.

Theorem 2 (Castigliano's principle). We introduce the additional strain energy functional

$$E^*(\sigma = \int_V A^*(\sigma)\,dV \tag{1.20}$$

Then there exists a single stress field providing the minimum of the functional $E^*(\sigma)$ on P, namely the field σ°, and the following equality is valid

$$\Phi(\mathbf{u}^\circ) = -E^*(\sigma^\circ) \tag{1.21}$$

Proof. From (1.12) the operator $\nabla A^*(\sigma)$ is coercive in $L_2(V)$, from which

[13] the existence and uniqueness of the minimum follows. Let us prove that the field σ° satisfies the identity

$$\int_V (\nabla A^*(\sigma^\circ), \sigma - \sigma^\circ)\, dV = 0 \tag{1.22}$$

for any $\sigma \in P$. In fact, from (1.17) – (1.19)

$$\int_V (\nabla A^*(\sigma^\circ), \sigma - \sigma^\circ)\, dV = \int_V (\varepsilon^\circ, \sigma - \sigma^\circ)\, dV =$$

$$\int_V f_i u_i^\circ dV + \int_{S_F} F_i u_i^\circ dS - \int_V (\varepsilon^\circ, \sigma^\circ)\, dV = 0$$

From [13], the identity (1.22) is a necessary and sufficient condition for the minimum. From (1.17), (1.11), (1.20) and (1.18)

$$\Phi(\mathbf{u}^\circ) = \int_V (-A^*(\sigma^\circ) + (\sigma^\circ, \varepsilon^\circ))\, dV - \int_V f_i u_i^\circ dV - \int_{S_F} F_i u_i^\circ dS =$$

$$-E^*(\sigma^\circ) + \int_V (\sigma^\circ, \varepsilon^\circ)\, dV - \int_V f_i u_i^\circ\, dV - \int_{S_F} F_i u_i^\circ dS = -E^*(\sigma^\circ)$$

Theorem 3. Let $\sigma \in P$ and $u \in H$, we set

$$\sigma(\mathbf{u}) \equiv \nabla A(\varepsilon(\mathbf{u})) \tag{1.23}$$

The identity

$$E^*(\sigma) - E^*(\sigma^\circ) + \Phi(\mathbf{u}) - \Phi(\mathbf{u}^\circ) = E^*(\sigma) - E^*(\sigma(\mathbf{u})) - \int_V (\varepsilon(\mathbf{u}), \sigma - \sigma(\mathbf{u}))\, dV \tag{1.24}$$

is valid.

Proof. Because of (1.21) it is sufficient to prove that

$$\Phi(\mathbf{u}) = -E^*(\sigma(\mathbf{u})) - \int_V (\varepsilon(\mathbf{u}), \sigma - \sigma(\mathbf{u}))\, dV$$

From (1.13), (1.11) and (1.16)

$$\Phi(\mathbf{u}) = -E^*(\sigma(\mathbf{u})) + \int_V (\varepsilon(\mathbf{u}), \sigma(\mathbf{u}))\, dV - \int_V f_i u_i dV -$$

$$\int_{S_F} F_i u_i dS = -E^*(\sigma(\mathbf{u})) + \int_V (\varepsilon(\mathbf{u}), \sigma(\mathbf{u}))\, dV - \int_V (\varepsilon(\mathbf{u}), \sigma)\, dV$$

this latter equality follows from (1.19).

Note 2. For the linear elasticity law

$$E^*(\sigma) - E^*(\sigma^\circ) = E^*(\sigma - \sigma^\circ), \quad \Phi(\mathbf{u}) - \Phi(\mathbf{u}^\circ) = E^*(\sigma(\mathbf{u}) - \sigma^\circ)$$

$$E^*(\sigma) - E^*(\sigma(\mathbf{u})) - \int_V (\varepsilon(\mathbf{u}), \sigma - \sigma(\mathbf{u}))\, dV = E^*(\sigma - \sigma(\mathbf{u}))$$

and the identity (1.24) agrees with the Prager-Synge relationship [1, 2]

$$E^*(\sigma - \sigma^\circ) + E^*(\sigma(\mathbf{u}) - \sigma^\circ) = E^*(\sigma - \sigma(\mathbf{u}))$$

and the identity (1.24) agrees with the Prager-Synge relationship [1, 2].

Note 3. The identity (1.24) is valid under weaker assumptions than (1.1), (1.2), for example it can be assumed that $A(\varepsilon) = A_1(\varepsilon_0) + A(\psi_\varepsilon)$, where $A_1(\tau)$, $A_2(\tau)$ are strictly convex continuously differentiable functions satisfying the conditions

$$\begin{gathered} A_i = dA_i / d\tau = 0, \ \tau = 0, \ i = 1, 2 \\ -a_i + b_i \tau^{p_i - 1} \leqslant dA_i / d\tau \leqslant c_i + d_i \tau^{p_i - 1}, \quad i = 1,2 \\ a_i, \ c_i \geqslant 0, \ b_i, \ d_i > 0 = \text{const}, \quad 1 < p_2 \leqslant p_1 \end{gathered} \tag{1.25}$$

For example the function

$$A(\varepsilon) = {}^9/_2\ K(\varepsilon_0)^2 + a_2(\psi_\varepsilon)^p, \quad 1 < p \leqslant 2$$

satisfies the constraints (1.25). Here $p_1 = 2$, $p_2 = p$.

The identity (1.24) is true even if $A(\varepsilon)$ is a continuously differentiable strictly convex function, $A = \nabla A = 0$, $\tau = 0$ and the following inequalities are satisfied

$$-a + b|\varepsilon|^p \leqslant (\nabla A(\varepsilon), \varepsilon) \leqslant c + d|\varepsilon|^p$$
$$a, \ c \geqslant 0, \ b, \ d \geqslant 0 = \text{const}, \ p > 1$$

Lemma 3. Let $\sigma \in P$, $\mathbf{u} \in \boldsymbol{H}$. Under the conditions (1.2), the estimates

$$\frac{1}{2\varkappa}\|\sigma - \sigma^\circ\|^2_{L_2(V)} + \frac{\gamma}{2}\|\varepsilon(\mathbf{u}) - \varepsilon^\circ\|^2_{L_2(V)} \leqslant \frac{1}{\gamma}\|\sigma - \sigma(\mathbf{u})\|^2_{L_2(V)} \tag{1.26}$$

can hold.

Proof. The inequalities

$$E^*(\sigma) - E^*(\sigma^\circ) \geqslant \frac{1}{2\varkappa}\|\sigma - \sigma^\circ\|^2_{L_2(V)} \tag{1.27}$$
$$\Phi(\mathbf{u}) - \Phi(\mathbf{u}^\circ) \geqslant \frac{\gamma}{2}\|\varepsilon(\mathbf{u}) - \varepsilon^\circ\|^2_{L_2(V)}$$

follow from (1.12), (1.3) and Theorem 10.4 [13]. Since the mappings $\nabla A(\varepsilon)$ and $\nabla A^*(\sigma)$ are reversible, then $\varepsilon(\mathbf{u}) = \nabla A^*(\sigma(\mathbf{u}))$, hence

$$E^*(\sigma) - E^*(\sigma(\mathbf{u})) - \int_V (\varepsilon(\mathbf{u}), \sigma - \sigma(\mathbf{u}))\,dV = \tag{1.28}$$

$$\int_0^1 \frac{d}{dt}E^*(\sigma(\mathbf{u}) + t(\sigma - \sigma(\mathbf{u})))\,dt - \int_V (\nabla A^*(\sigma(\mathbf{u})), \sigma - \sigma(\mathbf{u}))\,dV =$$

$$\int_0^1 dt \int_V \frac{1}{t}(\nabla A^*(\sigma(\mathbf{u}) + t(\sigma - \sigma(\mathbf{u}))) - \nabla A^*(\sigma(\mathbf{u})), t(\sigma - \sigma(\mathbf{u})))\,dV \leqslant$$

$$\int_0^1 \frac{1}{\gamma t}\|t(\sigma - \sigma(\mathbf{u}))\|^2_{L_2(V)}\,dt \leqslant \frac{1}{2\gamma}\|\sigma - \sigma(\mathbf{u})\|^2_{L_2(V)}$$

The inequality in (1.28) follows from (1.12), while (1.26) follows from (1.27), (1.28) and (1.24).

2. The plate equations. Let us consider the flexure of a rigidly clamped, constant-thickness plate, symmetrically loaded by a load applied to the upper and lower faces. The plate occupies the domain

$$V_h = \{(\mathbf{x}, x_3) \mid \mathbf{x} = (x_1, x_2), \ \ \mathbf{x} \in \Omega, \ \ -d < x_3 < d\}$$

Here Ω is the middle plane, l is its characteristic dimension, $2d$ is the plate thickness, $h = 2d / l$ is the characteristic relative thickness, then S_u is the side surface of the plate in (1.7)—(1.10), (1.14)—(1.6), $S_F = \Omega^+ \cup \Omega^-$, where Ω^+, Ω^- are the upper and lower faces of the plate

$$f_1 = f_2 = f_3 = 0, \quad F_1 = F_2 = 0 \quad \text{on} \quad S_F \tag{2.1}$$
$$F_3 = F^h / 2 \quad \text{on} \ \Omega^+, \qquad F_3 = -F^h / 2 \quad \text{on} \ \Omega^-$$

It is emphasized that the normal load F^h in (2.1) depends on h.

Let us derive the plate equations by using the Kirchhoff hypotheses. We introduce the notation: if $\mathbf{a} = \{a_{ij}\}$ is a tensor, $i, j = 1, 2, 3$, then $\mathbf{a} = \{\mathbf{a}_1, \mathbf{a}_2\}$, where

$$\mathbf{a}_1 \equiv (a_{11}, a_{12}, a_{21}, a_{22}), \quad \mathbf{a}_2 \equiv (a_{13}, a_{31}, a_{23}, a_{32}, a_{33}) \tag{2.2}$$

If the tensor $\mathbf{a}$ is provided with some superscript, then $\mathbf{a}_1$, $\mathbf{a}_2$ are provided with the same superscript, for example $\mathbf{a}_1{}^h$, $\mathbf{a}_2{}^h$ correspond to the tensor $\mathbf{a}^h = \{a_{ij}{}^h\}$ according to (2.2).

Let us write the elasticity law (1.6) in the form

$$\sigma_1 = \frac{\partial A(\varepsilon_1, \varepsilon_2)}{\partial \varepsilon_1} \equiv \left\{\frac{\partial A}{\partial \varepsilon_{11}}, \frac{\partial A}{\partial \varepsilon_{12}}, \frac{\partial A}{\partial \varepsilon_{21}}, \frac{\partial A}{\partial \varepsilon_{22}}\right\} \tag{2.3}$$

$$\sigma_2 = \frac{\partial A(\varepsilon_1, \varepsilon_2)}{\partial \varepsilon_2} \equiv \left\{\frac{\partial A}{\partial \varepsilon_{13}}, \frac{\partial A}{\partial \varepsilon_{31}}, \frac{\partial A}{\partial \varepsilon_{23}}, \frac{\partial A}{\partial \varepsilon_{32}}, \frac{\partial A}{\partial \varepsilon_{33}}\right\}$$

We set $\sigma_2 = 0$ in (2.3) according to the Kirchhoff hypotheses, and analyze (2.3) for fixed ε_1 as a system of equations in ε_2. Using (1.3) and (1.4) we can show that this system is uniquely solvable for any ε_1 and its solution $\varepsilon_2(\varepsilon_1)$ has the form

$$\varepsilon_2(\varepsilon_1) = (0, 0, 0, 0, Q(\varepsilon_1)) \tag{2.4}$$

where $Q(\varepsilon_1)$ is continuous and satisfies the constraints

$$|Q(\varepsilon_1)| \leqslant \varkappa / \gamma \, |\varepsilon_1| \tag{2.5}$$

$$(\varepsilon_1{}^1, \varepsilon_1) \equiv \sum_{i,j=1}^{2} \varepsilon_{ij}{}^1 \varepsilon_{ij}, \quad |\varepsilon_1| \equiv (\varepsilon_1, \varepsilon_1)^{1/2}$$

Substituting (2.4) into $A(\varepsilon)$, we obtain a function of four real arguments

$$D(\varepsilon_1) \equiv A(\varepsilon_1, 0, 0, 0, 0, Q(\varepsilon_1)) \tag{2.6}$$

We use the Kirchhoff kinematic hypotheses according to which

$$u_i(\mathbf{x}, x_3) = -w_{,i}(\mathbf{x})\, x_3, \quad i = 1, 2, \quad u_3(\mathbf{x}, x_3) = w(\mathbf{x}) \tag{2.7}$$

Substituting (2.7) into (1.5), we obtain

$$\varepsilon_1(w) = -\mu(w)\, x_3, \quad \mu(w) \equiv \left(\frac{\partial^2 w}{\partial x_1{}^2}, \frac{\partial^2 w}{\partial x_1 \partial x_2}, \frac{\partial^2 w}{\partial x_2 \partial x_1}, \frac{\partial^2 w}{\partial x_2{}^2}\right) \tag{2.8}$$

Replacing $A(\varepsilon(\mathbf{u}))$ in (1.16) by $D(\varepsilon_1(w))$ and substituting the hypothesis (2.7) into the linear part of (1.16) by taking account of (2.1), we obtain the total energy functional of a thin plate

$$\Psi_h(w) = \int_{V_h} D(-\mu(w)\, x_3)\, d\mathbf{x} dx_3 - \int_{\Omega} F^h w \, d\mathbf{x}$$

Lemma 4. $D(\varepsilon_1)$ is twice continuously differentiable, strictly convex and for any ε_1, $\varepsilon_1{}^1$ the following estimates are valid:

$$\gamma\, |\varepsilon_1 - \varepsilon_1{}^1|^2 \leqslant (\nabla D(\varepsilon_1) - \nabla D(\varepsilon_1{}^1), \; \varepsilon_1 - \varepsilon_1{}^1) \leqslant \varkappa\, |\varepsilon_1 - \varepsilon_1{}^1|^2 \tag{2.9}$$

$$\nabla D(\varepsilon_1) \equiv \{\partial D / \partial \varepsilon_{11}, \partial D / \partial \varepsilon_{12}, \partial D / \partial \varepsilon_{21}, \partial D / \partial \varepsilon_{22}\}$$

$$(\varepsilon_1 = \{\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{21}, \varepsilon_{22}\}, \quad \varepsilon_1{}^1 = \{\varepsilon_{11}{}^1, \varepsilon_{12}{}^1, \varepsilon_{21}{}^1, \varepsilon_{22}{}^1\})$$

Also valid are the equalities

$$D(-\varepsilon_1) = D(\varepsilon_1) \tag{2.10}$$

$$\nabla D(\varepsilon_1) = \partial A(\varepsilon_1, 0, 0, 0, 0, Q(\varepsilon_1)) / \partial \varepsilon_1 \tag{2.11}$$

Let Γ be the boundary of Ω; we introduce the space of admissible deflections

$$W_2^{2,0}(\Omega) = \{w \mid w \in W_2^2(\Omega).\ \ w = \partial w / \partial n = 0 \text{ on } \ \}$$

Theorem 4. There exists a unique function $w^h \in W_2^{2,0}(\Omega)$ which provides the minimum of the functional $\Psi_h(w)$ on $W_2^{2,0}(\Omega)$.

Because of Lemma 4, the proof follows from [14].

We introduce the notation

$$\delta^h = \{\delta_{ij}^h\}, \quad \delta_1^h \equiv \varepsilon_1(w^h) = -\mu(w^h)x_3, \quad \delta_2^h \equiv \varepsilon_2(\delta_1^h) = (0, 0, 0, 0, Q(\delta_1^h)) \tag{2.12}$$

$$\alpha^h = \{\alpha_{ij}^h\}, \quad \alpha_1^h \equiv \nabla D(\delta_1^h), \quad \alpha_2^h \equiv 0 \tag{2.13}$$

According to (2.12), (2.13), the tensors δ^h, α^h yield the strain and stress in the plate equations. There follows from (2.11), (2.13) and the definition (2.4) that

$$\alpha_1^h = \partial A(\delta_1^h, \delta_2^h) / \partial \varepsilon_1, \ \alpha_2^h = 0 = \partial A(\delta_1^h, \delta_2^h) / \partial \varepsilon_2$$

i.e.,

$$\alpha^h = \nabla A(\delta^h) \tag{2.14}$$

3. Error estimates of the plate equations. The idea for obtaining the estimates borrowed from [3, 4], is that two fields are constructed by means of the solution w^h, δ^h, α^h of the plate equations: the statically admissible stress field $\sigma \in P$ and the kinematically admissible displacement field $\mathbf{u} \in H$, where the norm of the difference $\sigma - \sigma(\mathbf{u})$ will admit of an explicit estimate in terms of the parameter h and the derivatives of w^h, and moreover, (1.26) is applied.

Let us make the change of variable $x_3 = hz$, then the domain V_h is mapped into the domain $V_1 = \{(\mathbf{x}, z) \mid \mathbf{x} \in \Omega, \ -l/2 < z < l/2\}$. If the function φ is given in V_h, then we shall denote the function g given in V_1 by the formula $g(\mathbf{x}, z) = \varphi(\mathbf{x}, hz)$ by the same symbol φ. Evidently

$$\|\varphi\|_{L_2(V_h)} = h^{1/2}\|\varphi\|_{L_2(V_1)} \tag{3.1}$$

The final estimates will be obtained in the norm of $L_2(V_1)$ (and not $L_2(V_h)$), since the norm in $L_2(V_1)$ does not depend on h.

The function w^h satisfies an identity for any $w \in W_2^{2,0}(\Omega)$

$$\int_{V_h} (\nabla D(-\mu(w^h)x_3), -\mu(w)x_3)\, d\mathbf{x}dx_3 = \int_\Omega F^h w\, d\mathbf{x} \tag{3.2}$$

Substituting $w = w^h$ into (3.2), using (2.9) for $\varepsilon_1^1 = 0$ and integrating (3.2) with respect to x_3, we obtain

$$\frac{2}{3}\gamma^2 d^3 \int_\Omega (\mu(w^h), \mu(w^h))\, d\mathbf{x} \leqslant \int_\Omega F^h w^h d\mathbf{x} \tag{3.3}$$

The estimate

$$\|w^h\|_{W_2^2(\Omega)} \leqslant \frac{c}{\gamma h^3}\|F^h\|_{L_2(\Omega)} \tag{3.4}$$

follows from (3.3) and from the imbedding theorems [15]. Here and henceforth the symbol c denotes different constants which are independent of h.

The estimates

$$\|\delta^h\|_{L_2(V_1)} \leqslant \frac{c}{\gamma h^2}\|F^h\|_{L_2(\Omega)}, \ \|\alpha^h\|_{L_2(V_1)} \leqslant \frac{c\varkappa}{\gamma h^2}\|F^h\|_{L_2(\Omega)} \tag{3.5}$$

follow from the definitions (2.12), (2.13) and (3.4), (2.5), (2.14), (1.4), (3.1). Let F^h be representable as $F^h = h^2 q^h$, and

$$\| q^h \|_{L_2(\Omega)} \leqslant c_q, \quad c_q = \text{const} \tag{3.6}$$

Then estimates with constants independent of h follow from (3.4) — (3.6)

$$\| w^h \|_{W_2^2(\Omega)} \leqslant \frac{c}{\gamma h} c_q$$

$$\| \delta^h \|_{L_2(V_1)} \leqslant \frac{c}{\gamma} c_q, \quad \| \alpha^h \|_{L_2(V_1)} \leqslant \frac{c\varkappa}{\gamma} c_q$$

We propose more, namely, let $w^h \in C^4(\Omega^c)$ (Ω^c is the closure of the domain Ω), δ^h, $\alpha^h \in C^2(V_h{}^c)$ and estimates with constant c_F independent of h are valid

$$\| w^h \|_{C^4(\Omega^c)} \leqslant \frac{c_F}{\gamma h} \tag{3.7}$$

$$| \delta_{ij}{}^h | \leqslant \frac{c_F}{\gamma}, \quad | \delta^h_{ij,n} | \leqslant \frac{c_F}{\gamma}, \quad \left| \frac{\partial^2 \delta_{ij}{}^h}{\partial x_n \partial x_m} \right| \leqslant \frac{c_F}{\gamma}; \tag{3.8}$$

$i, j = 1, 2, 3, \quad n, m = 1, 2$

$$| \alpha_{ij}{}^h | \leqslant \frac{\varkappa c_F}{\gamma}, \quad | \alpha^h_{ij,n} | \leqslant \frac{\varkappa c_F}{\gamma}, \quad \left| \frac{\partial^2 \alpha_{ij}{}^h}{\partial x_n \partial x_m} \right| \leqslant \frac{\varkappa c_F}{\gamma}; \quad i, j, n, m = 1, 2 \tag{3.9}$$

Theorem 5. Let $\Gamma \in C^\infty$, $A(\varepsilon)$ satisfies (1.1), (1.2) and $A_2(\psi_\varepsilon)$ as a function of the nine real arguments $\varepsilon_{ij}{}'$ is quadruply continuously differentiable with respect to $\varepsilon_{ij}{}'$ and the following estimates are valid

$$\left| \frac{\partial^3 A_2(\psi_\varepsilon)}{\partial \varepsilon'_{ij} \partial \varepsilon'_{ks} \partial \varepsilon'_{mn}} \right| \leqslant \frac{M_1}{1 + \psi_\varepsilon}, \quad i, j, k, s, m, n = 1, 2, 3, \quad M_1 = \text{const}$$

$$\left| \frac{\partial^4 A_2(\psi_\varepsilon)}{\partial \varepsilon'_{ij} \partial \varepsilon'_{ks} \partial \varepsilon'_{mn} \partial \varepsilon'_{pq}} \right| \leqslant M_2; \quad i, j, k, s, m, n, p, q = 1, 2, 3, \quad M_2 = \text{const}$$

$$\| q^h \|_{C^3(\Omega^c)} \leqslant c_q, \quad c_q = \text{const}$$

Then w^h, δ^h, α^h satisfy the constraints (3.7) — (3.9).

The proof is not presented because of its awkwardness, we just note some fundamental points.

From (3.2) the function w^h is a generalized solution of a quasi-linear fourth order differential equation in the domain Ω

$$\sum_{i,j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j} \left[\int_{-d}^{a} - \frac{\partial D(-\mu(w) x_3)}{\partial \varepsilon_{ij}} x_3 dx_3 \right] = F^h \tag{3.10}$$

$$w = \partial w / \partial n = 0 \quad \text{on } \Gamma$$

Sufficient condition for which the generalized solution of (3.10) is regular are known from [12, 16, 17]; using the conditions of Theorem 5, it can be shown that (3.10) satisfies these sufficient conditions, from which we obtain that w^h is a classical solution of (3.10), i.e. $w^h \in C^4(\Omega^c)$, and satisfies (3.10); furthermore, using the definition of the fields δ^h, α^h and the conditions of Theorem 5, we can prove (3.7) — (3.9).

An example of the elasticity law satisfying the conditions of Theorem 5 is

$$A_2(\psi_\varepsilon) = 2G_1(\psi_\varepsilon)^2[1 + G_2(\psi_\varepsilon)^2(1 + G_1(\psi_\varepsilon)^2)^{-1}]$$

Henceforth, compliance with (3.7) — (3.9) is assumed.

We construct a statically admissible field $\sigma = \{\sigma_{ij}\}$ "close" to α^h. We set

$$\sigma_{ij} = \alpha_{ij}^h, \quad i, j = 1, 2, \quad \sigma_{i3} = \sigma_{3i} = -\int_{-d}^{x_3} \sum_{j=1}^{2} \sigma_{ij,j}(\mathbf{x}, \tau)\, d\tau, \quad i = 1, 2 \tag{3.11}$$

$$\sigma_{33} = -\int_0^{x_3} \sum_{i=1}^{2} \sigma_{3i,i}(\mathbf{x}, \tau)\, d\tau \tag{3.12}$$

The stresses σ_2 are obtained from σ_1 by integrating the equilibrium equation (1.7). We verify that the field σ satisfies the boundary conditions (1.8) on Ω^+, Ω^-. From (2.10), (2.12), (2.13) the stresses σ_1 are odd in x_3, hence $\sigma_{i3} = \sigma_{3i}$ are even in x_3, $i = 1, 2$, and $\sigma_{i3} = \sigma_{3i} = 0$ on Ω^+, Ω^-, $i = 1, 2$ from (3.11).

The stress σ_{33} is odd in x_3, hence

$$2\sigma_{33}(\mathbf{x}, d) = -\int_{-d}^{d} \sum_{i=1}^{2} \sigma_{3i,i}(\mathbf{x}, \tau)\, d\tau =$$

$$\int_{-d}^{d} (d - x_3) \sum_{i,j=1}^{2} \frac{\partial^2 \sigma_{ij}(\mathbf{x}, x_3)}{\partial x_i \partial x_j}\, dx_3 =$$

$$\sum_{i,j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j} \left[\int_{-d}^{d} -\frac{\partial D(-\mu(w^h) x_3)}{\partial \varepsilon_{ij}} x_3 dx_3 \right] = F^h$$

from which $\sigma_{33}(\mathbf{x}, \pm d) = \pm F^h / 2$, therefore, $\sigma \in P$.

From (3.11), (3.12), (3.9) we obtain (mes V is the Euclidean measure of the domain V)

$$\| \sigma - \alpha^h \|_{L_2(V_h)} = \| \sigma_2 \|_{L_2(V_h)} \leqslant \frac{\varkappa c_F h}{\gamma} (\operatorname{mes} V)^{1/2} \leqslant \frac{\varkappa c_F (\operatorname{mes} \Omega)^{1/2}}{\gamma} h^{3/2} \tag{3.13}$$

We construct the field $\mathbf{u} \in H$ so that $\varepsilon(\mathbf{u})$ "differs slightly" from δ^h, namely, let us set

$$u_i(\mathbf{x}, x_3) = -w_{,i}^h(\mathbf{x})\, x_3, \quad i = 1, 2, \tag{3.14}$$

$$u_3(\mathbf{x}, x_3) = w^h(\mathbf{x}) + \eta_\rho(\mathbf{x})\, \vartheta^h(\mathbf{x}, x_3)$$

$$\vartheta^h(\mathbf{x}, x_3) = \int_0^{x_3} \delta_{33}^h(\mathbf{x}, \tau)\, d\tau \tag{3.15}$$

Here $\eta_\rho(\mathbf{x})$ is a smooth function finite in Ω, equal to unity in the subdomain Ω_ρ (Ω_ρ consists of points at a distance more than ρ from the boundary) and such that

$$0 \leqslant \eta_\rho(\mathbf{x}) \leqslant 1, \quad | \nabla \eta_\rho(\mathbf{x}) | \leqslant c / \rho \tag{3.16}$$

According to condition (3.8) δ_{33}^h is continuously differentiable with respect to x_1, x_2, hence $\mathbf{u} \in H$ and it follows from (3.14), (3.15), (2.12) and (2.8) that:

$$\varepsilon_{ij}(\mathbf{u}) = \delta_{ij}^h, \quad i, j = 1, 2, \quad \varepsilon_{i3}(\mathbf{u}) = \varepsilon_{3i}(\mathbf{u}) = \eta_{\rho,i}\vartheta^h + \eta_\rho \vartheta_{,i}^h, \quad i = 1, 2$$

$$\varepsilon_{33}(\mathbf{u}) = \eta_\rho(\mathbf{x})\, \delta_{33}^h$$

From (3.8) and (3.15), $|\vartheta^h| \leqslant hlc_F/\gamma$, $|\vartheta_{,i}^h| \leqslant hlc_F/\gamma$, $i=1,2$, then we obtain because of (3.16)

$$|\varepsilon_{i3}(\mathbf{u})| \leqslant hcc_F\gamma^{-1}(\rho^{-1}+1), \quad \mathbf{x} \notin \Omega_\rho, \quad |\varepsilon_{i3}(\mathbf{u})| \leqslant c_F\gamma^{-1}$$
$$\mathbf{x} \in \Omega_\rho, \quad i=1,2$$

Hence

$$\|\varepsilon_{i3}(\mathbf{u})\|_{L_2(V_h)} \leqslant \frac{cc_F}{\gamma}(\rho^{-1}+1)h^{3/2}\rho^{1/2} + h^{3/2}\frac{c_F}{\gamma}(l \text{ mes }\Omega)^{1/2} \qquad (3.17)$$

The functions $\varepsilon_{33}(\mathbf{u})$ and δ_{33}^h agree for $\mathbf{x} \in \Omega_\rho$, hence we obtain from (3.8)

$$\|\varepsilon_{33}(\mathbf{u}) - \delta_{33}^h\|_{L_2(V_h)} \leqslant \frac{cc_F}{\gamma}h^{1/2}\rho^{1/2} \qquad (3.18)$$

We set $\rho = h$, then it follows from (3.17) and (3.18) that

$$\|\boldsymbol{\varepsilon}(\mathbf{u}) - \boldsymbol{\delta}^h\|_{L_2(V_h)} = \Big(2\sum_{i=1}^{2}\|\varepsilon_{i3}(\mathbf{u})\|^2_{L_2(V_h)} + \|\varepsilon_{33}(\mathbf{u}) - \delta_{33}^h\|^2_{L_2(V_h)}\Big)^{1/2} \leqslant \frac{cc_F}{\gamma}h \qquad (3.19)$$

It also follows from (3.19),(2.14) and (1.4) that

$$\|\boldsymbol{\sigma}(\mathbf{u}) - \boldsymbol{\alpha}^h\|_{L_2(V_h)} \leqslant \frac{c\varkappa c_F}{\gamma}h \qquad (3.20)$$

and from (3.13) and (3.20) that

$$\|\boldsymbol{\sigma}(\mathbf{u}) - \boldsymbol{\sigma}\|_{L_2(V_h)} \leqslant \frac{c\varkappa c_F}{\gamma}h \qquad (3.21)$$

Using (3.21), (1.26) and (3.1), we obtain the final error estimates of the solution of the plate equations (as before, $\boldsymbol{\varepsilon}^\circ$, $\boldsymbol{\sigma}^\circ$ is the solution of the spatial problem of elasticity)

$$\|\boldsymbol{\sigma}^\circ - \boldsymbol{\alpha}^h\|_{L_2(V_1)} \leqslant c\left(\frac{\varkappa}{\gamma}\right)^{3/2} c_F h^{1/2}$$

$$\|\boldsymbol{\varepsilon}^\circ - \boldsymbol{\delta}^h\|_{L_2(V_1)} \leqslant c\frac{\varkappa}{\gamma^2}c_F h^{1/2}$$

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ANALYTICAL INVESTIGATION OF THE UNLOADING WAVE

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In the general case, the determination of the unloading wave shape [1] in the theory of elastic plastic wave propagation is reduced to the solution of a functional equation of complex structure. A characteristics method [2] is proposed for the approximate construction of the unloading wave, in particular loading cases formulas are obtained to determine its initial slope [3] and the next derivatives at the initial point [4 — 6]. An investigation of the general properties of an unloading wave is given in [7]. It is shown that as the load tends to zero asymptotically, the unloading wave at the end of a semi-infinite bar has an asymptote with a slope determined by the elastic wave velocity.

An investigation of the functional equation is given in this paper and a method of solution of this equation in the form of a power series is proposed. This approach to the problem permits obtaining both known and some new results. In the general loading case, formulas are obtained to determine the initial slope of the unloading wave and a method of determining the next derivatives at the initial point is indicated. Conditions are found for linear hardening for which the unloading wave is a straight line. The existence of an asymptote different from those mentioned in [7] is proved; it is shown how to continue the solution to adjacent sections by means of some known section, and an unloading wave in a material with delayed yielding is investigated.